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P-Prime and Small P-Prime Ideals in Near-Rings

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ABSTRACT

In this paper we introduce the notions of P-prime ideal and small P-prime ideal which is a generalization of prime ideal and obtain equivalent conditions for an ideal to be P-prime. Also we introduce the notions of P-m-system and small P-m-system in near-rings.

Key words: P-prime ideal, small P-prime ideal, P-m-system, small P-m-system.

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1. Introduction

Anderson and Fuller [2] defined small right ideal in rings. Aldosary and Bashammakh [1] obtained results on small right ideal satisfying the chain conditions. For the basic terminology and notation we refer to [3].

Throughout this paper, N denotes a zero-symmetric near-ring and P denotes an ideal of N which is arbitrary but fixed. For any subset A of N, $< A >$ denotes the ideal of N generated by A. For any a $\in$ N, $< a >$ stands for the ideal of N generated by a. An ideal A of N is said to be prime if BC $\subseteq$ A implies B $\subseteq$ A or C $\subseteq$ A for ideals B, C of N. A non-empty subset M of N is said to be an m-system if for any a, b $\in$ M there exists $a_1 \in < a >$ and $b_1 \in < b >$ such that $a_1 b_1 \in$ M.

In this paper we introduce the notions of P-prime ideal, small prime and small P-prime ideal which is a generalization of prime ideal and obtain equivalent conditions for an ideal to be P-prime. In this paper we extend the small right ideal in rings to the small right ideal in near-rings. Also we introduce the notions of P-m-system, small m-system and small P-m-system in near-rings.

2. P-prime ideals

Now we introduce the notion of P-prime ideal in near-rings.

Definition 2.1 An ideal A of N is said to be P-prime if BC + P $\subseteq$ A implies B $\subseteq$ A or C $\subseteq$ A for ideals B, C of N.

If A is a prime ideal then clearly A is a P-prime ideal for any ideal P. Now we give an example of a P-prime ideal that is not prime.

Example 2.2 Let N = {0, a, b, c} be the Klein four group. Define multiplication in N as follows.

	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then (N, +, .) is a near-ring (see Pilz [3], p. 407, scheme 7). Here the ideals are {0}, {0, a}, {0, b} and N. Let P = {0, b}. Clearly {0} is P-prime but not prime since {0, a} {0, b} ⊆ {0} but {0, a} ⊈ {0} and {0, b} ⊈ {0}.

Theorem 2.3 Let A be an ideal of N. Then the following are equivalent:

(1) A is a P-prime ideal.

(2) $< BC >$+ P ⊆ A implies B ⊆ A or C ⊆ A for ideals B, C of N.

(3) b ∉ A and c ∉ A implies $< b >$ $< c >$ + P ⊈ A for all b, c ∈ N.

(4) B ⊃ A and C ⊃ A implies BC + P ⊈ A for ideals B, C of N.

(5) B ⊈ A and C ⊈ A implies BC + P ⊈ A for ideals B, C of N.

Proof. (4) ⇒ (5) Assume that B and C are ideals of N such that B ⊈ A and C ⊈ A. Let b ∈ B\A and c ∈ C\A. Then $< b >$ + A ⊃ A and $< c >$ + A ⊃ A. Thus ($< b >$ + A)($< c >$ + A) + P ⊈ A. So there exists $b_1 \in < b >$, $c_1 \in < c >$, a_1, a_2 ∈ A and p_1 ∈ P such that $((b_1+ a_1)(c_1+a_2))+p_1 \notin A$. Then $b_1 (c_1+a_2) - b_1 c_1+ b_1 c_1+p_2+a_1(c_1+a_2) \notin A$ for some p_2 ∈ P. Since $b_1 (c_1 + a_2) - b_1 c_1 \in A$ and $a_1(c_1 + a_2) \in A$, we have $b_1c_1 + p_2 \notin A$. Hence BC + P ⊈ A. The other implications are straightforward.

Definition 2.4 A non-empty subset M of N is said to be an P-m-system if for any a, b ∈ M there exists $a_1 \in < a >$, $b_1 \in < b >$ and p ∈ P such that $a_1b_1 + p \in M$.

Clearly every m-system is a P-m-system.

Lemma 2.5 Let A be an ideal of N. Then A is a P-prime ideal if and only if N\A is an P-m-system.

Proof. Let A be an ideal. Assume that A is a P-prime ideal. Let b, c ∈ N\A. Since A is P-prime, by (3) of Theorem 2.3, $< b > < c > +$ P $\nsubseteq$ A implies there exists $b_1 \in < b >$, $c_1 \in < c >$ and $p_1 \in P$ such that $b_1c_1 + p_1 \notin A$. Thus $b_1c_1 + p_1 \in N\backslash A$. Hence N\A is an P-m-system.

Conversely, assume that for any x, y ∈ N such that x ∉ A and y ∉ A. Since N\A is an P-m-system, there exists $x_1 \in < x >$, $y_1 \in < y >$ and p ∈ P such that $x_1y_1 + p \in N\backslash A$. Then $< x > < y > +$ P $\nsubseteq$ A. Hence A is a P-prime ideal.

Theorem 2.6 Let M ⊆ N be a non-void P-m-system in N and I an ideal of N with I ∩ M = ∅. Then I is contained in a P-prime ideal A ≠ N with A ∩ M = ∅.

Proof. Let $\mathbb{F} = \{J \mid J \text{ is an ideal of } N, J \supseteq I \text{ and } J \cap M = \emptyset\}$. Clearly I ∈ 𝔽. By Zorn's Lemma, 𝔽 contains a maximal element A. Thus A is an ideal and A is not equal to N. Assume B and C are ideals of N such that B ⊃ A and C ⊃ A. Take some b ∈ B ∩ M and c ∈ C ∩ M. Now $< b > < c > \subseteq$ BC and there exists $b_1 \in < b >$, $c_1 \in < c >$ and p ∈ P such that $b_1c_1 + p \in M$. So $(< BC > + P) \cap M \neq \emptyset$, $< BC > + P \nsubseteq A$ and BC + P $\nsubseteq$ A.

The following is an immediate Corollary of Theorem 2.6.

Corollary 2.7 ([3], Proposition 2.81) Let M ⊆ N be a non-void m-system in N and I an ideal of N with I ∩ M = ∅. Then I is contained in a prime ideal A ≠ N with A ∩ M = ∅.

Let P(I) denote the prime radical of I which is the intersection of all prime ideals containing I and ℙ(I) denote the P-prime radical of I which is the intersection of all P -prime ideals containing I.

Lemma 2.8 If n ∈ ℙ (I), then there exists a positive integer k such that $n^k \in I$.

Proof. If n ∈ N, then the set $M = \{n, n^2, n^3, \ldots\}$ is an P-m-system. If I ∩ M = ∅ then by Theorem 2.6, there is some P-prime ideal A ⊇ I with A ∩ M = ∅, a contradiction to n ∈ ℙ(I). Hence I ∩ M ≠ ∅ and there exists a positive integer k such that $n^k \in I$.

We have the following corollary from Lemma 2.8.

Corollary 2.9 If n ∈ P(I), then there exists a positive integer k such that $n^k \in I$.

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3. Small P-prime ideals

Now we introduce the notion of small right ideal in near-rings similar to the notion of small right ideal in rings (see [2], P.72).

Definition 3.1 An ideal A of N is called a small right ideal if for any right ideal B of N the equation A + B = N implies B = N.

In a near-ring, {0} is always a small right ideal but N is not a small right ideal.

Lemma 3.2 The sum of two small right ideal in N is again a small right ideal in N.

Proof. Let A and B be small right ideals in N such that (A + B) + C = N for any right ideal C of N. Now A + (B + C) = (A + B) + C = N. Since A is a small right ideal in N, we have B + C = N. Since B is a small right ideal in N, we have C = N. Thus A + B is a small right ideal in N.

Definition 3.3 An ideal A of N is said to be small prime if BC ⊆ A implies B ⊆ A or C ⊆ A for small right ideals B, C of N.

Clearly every prime ideal is a small prime ideal but the converse need not be true as the following example shows.

Example 3.4 In Example 2.2, the ideals are {0}, {0, a}, {0, b} and N and the small right ideal is {0}. Let A = {0}. Then A is small prime but not prime since {0, a} {0, b} = {0} ⊆ A but {0, a} ⊄ A and {0, b} ⊄ A.

Notation 3.5 Let a ∈ N. Let $<a>_s$ denote a small right ideal of N generated by a.

Definition 3.6 A non-empty subset M of N is said to be a small m-system if for any a, b ∈ M, whenever $<a>_s$ and $<b>_s$ exist then there exists $a_1 \in <a>_s$ and $b_1 \in <b>_s$ such that $a_1b_1 \in$ M.

Every m-system is a small m-system, but the converse need not be true. In Example 2.2, the only small right ideal is {0}. Let M = {a, b}. Then M is a small m-system, but not an m-system since for a, b ∈ M there is no $a_1 \in <a>$ and $b_1 \in <b>$ such that $a_1b_1 \in$ M.

Lemma 3.7 An ideal A of N is small prime if and only if N\A is a small m-system.

Proof. Straightforward.

Definition 3.8 An ideal A of N is said to be small P-prime if BC+P ⊆ A implies B ⊆ A or C ⊆ A for small right ideals B, C of N.

If A is a small prime ideal then clearly A is a small P-prime ideal for any ideal P. Now we give an example of a small P-prime ideal that is not small prime.

Example 3.9 Let N = $\{0, a, b, c\}$ be the Klein four group. Define multiplication in N as follows.

.	0	a	b	c
0	0	0	0	0
a	0	0	0	a
b	0	a	b	b
c	0	a	b	c

Then (N, +, .) is a near-ring (see Pilz [3], p. 407, scheme 8). Here the ideals are {0}, {0, a} and N and the small right ideals are {0} and {0, a}. Let P = {0, a}. Clearly {0} is small P-prime but not small prime since {0, a} {0, a} ⊆ {0} but {0, a} ⊈ {0}.

Definition 3.10 A non-empty subset M of N is said to be a small P-m-system if for any a, b ∈ M, whenever $<a>_s$ and $<b>_s$ exist then there exists $a_1 \in <a>_s$, $b_1 \in <b>_s$ and p ∈ P such that $a_1b_1 + p \in M$.

Clearly every small m-system is a small P-m-system.

Theorem 3.11 Let M ⊆ N be a non-void small P-m-system in N and I be a small right ideal of N with I ∩ M = ∅. Then I is contained in a small P-prime ideal A ≠ N with A ∩ M = ∅.

Proof. Let 𝕍 = {J | J is a small right ideal of N and J ∩ M = ∅}. Since I ∈ 𝕍, 𝕍 is non-empty. Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be an arbitrary chain of small right ideals in 𝕍. Let 𝕂 = $\cup A_i$. Clearly 𝕂 is an ideal of N. Suppose for any right ideal Q of N, 𝕂 + Q = N implies there exists k ∈ 𝕂 and q ∈ Q such that k +q = 1. Since 𝕂 = $\cup A_i$, we have k ∈ A_i for some small right ideal Ai ∈ 𝕍. So A_i +Q = N implies Q = N. Hence 𝕂 is a small right ideal of N. By the definition of 𝕍, 𝕂 ∩ M = ∅. By Zorn's Lemma, 𝕍 contains a maximal element say A. Thus A is a small right ideal of N such that A ∩ M = ∅. Assume that B and C are small right ideals of N such that BC +P ⊆ A. Suppose B ⊈ A and C ⊈ A. Let b ∈ B such that b ∉ A and c ∈ C such that c ∉ A. Then A ⊂ $<b>_s$ +A and A ⊂ $<c>_s$ +A. By Lemma 3.2, $<b>_s$ +A and $<c>_s$ +A are small right ideals. By the maximality of A, $(\langle b\rangle_s + A) \cap M \neq \emptyset$ and $(\langle c\rangle_s + A) \cap M \neq \emptyset$. Let $x \in (\langle b\rangle_s + A) \cap M$ and $y \in (\langle c\rangle_s + A) \cap M$. Since $\langle x\rangle_s$ and $\langle y\rangle_s$ exists and M is a small P-m-system, there exists $x_1 \in \langle x\rangle_s$, $y_1 \in \langle y\rangle_s$ and $p \in P$ such that $x_1 y_1 + p \in M$. Also $x_1 y_1 + p \in ((\langle b\rangle_s + A)(\langle c\rangle_s + A)) + P$. So there exists $b_1 \in \langle b\rangle_s$, $c_1 \in \langle c\rangle_s$, $p_1 \in P$ and $a_1, a_2 \in A$ such that $x_1 y_1 + p = (b_1 + a_1)(c_1 + a_2) + p_1 = b_1 (c_1 + a_2) + a_1 (c_1 + a_2) + p_1 = (b_1(c_1 + a_2) - b_1 c_1) + b_1 c_1 + a_1 (c_1 + a_2) + p_1 = (b_1(c_1 + a_2) - b_1 c_1) + b_1 c_1 + p_2 + a_1 (c_1 + a_2)$ for some $p_2 \in P$ implies $x_1 y_1 + p \in A$. Thus $A \cap M \neq \emptyset$, a contradiction. Then $B \subseteq A$ or $C \subseteq A$. Hence A is a small P-prime ideal of N.

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